# General Polynomial Solution

**1.** $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-c)(c-a)}+\dfrac{1}{(c-a)(a-b)}=\dfrac{(c-a)+(a-b)+(b-c)}{(a-b)(b-c)(c-a)}=0$

$\left(\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}\right)^2=\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}+2\left[\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-c)(c-a)}+\dfrac{1}{(c-a)(a-b)}\right]$

$=\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$ , by the above. $\therefore\sqrt{\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}}=\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}$

**2.** Let $p(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ , then by Division Algorithm, $p(x) = (x - k)\, q(x) + r(x)$.

Since $\deg [r(x)] < \deg [(x - k)] = 1$, therefore $r(x) = c$, a constant. $\therefore$ $p(x) = (x - k)\, q(x) + c$.

Now, $p(k) = (k - k)\, q(x) + c = a_0k^3 + a_1k^2 + a_2k + a_3 = 0 \Rightarrow c = 0$. $\therefore$ $p(x) = (x - k)\, q(x)$.

$\therefore$ $x - k$ is a factor of $p(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ .

Let $F(x, y, z) = x^3 (y - z) + y^3 (z - x) + z^3 (x - y)$.

Then $F(y, y, z) = x^3 (y - z) + y^3 (z - y) + z^3 (y - y) = x^3 (y - z) - y^3 (y - z) = 0$

$\therefore$ $(x - y)$ is a factor of $F(x, y, z)$ by Factor Theorem.

Since $F(x, y, z)$ is cyclic, $(x - y)(y - z)(z - x)$ is a factor of $F(x, y, z)$ .

Deg $[F(x, y, z)] = 4$, $\deg[(x - y)(y - z)(z - x)] = 3$ , the remaining factor must be of degree 1 : $k(x + y + z)$.

$F(x, y, z) = k(x + y + z)(x - y)(y - z)(z - x)$ …..(1)

Comparing coefficients of $x^3y$ on both sides of (1), we have $k = -1$.

$\therefore$ $F(x, y, z) = -(x + y + z)(x - y)(y - z)(z - x) = (x + y + z)(y - x)(z - y)(z - x)$

**3.** **(a)** Let $F(a, b, c) = (b + c)^3 (b - c) + (c + a)^3 (c - a) + (a + b)^3 (a - b)$.

$F(b, b, c) = (b + c)^3 (b - c) + (c + b)^3 (c - b) + (a + b)^3 (b - b) = (b + c)^3 (b - c) - (c + b)^3 (b - c) = 0$

$\therefore$ $(a - b)$ is a factor of $F(a, b, c)$ by Factor Theorem.

Since $F(a, b, c)$ is cyclic, $(a - b)(b - c)(c - a)$ is a factor of $F(a, b, c)$ .

Deg $[F(a, b, c)] = 4$, $\deg[(a - b)(b - c)(c - a)] = 3$ , the remaining factor must be of deg 1 : $k(a + b + c)$.

$F(a, b, c) = k(a + b + c)(a - b)(b - c)(c - a)$ …..(1)

Comparing coefficients of $a^3b$ on both sides of (1), we have $k = -2$.

$F(a, b, c) = -2(a + b + c)(a - b)(b - c)(c - a) = 2(a + b + c)(b - a)(c - b)(a - c)$ .

**(b)** Let $F(a, b, c) = (b - c)(c + a - b)(a + b - c) + (c - a)(a + b - c)(b + c - a) + (a - b)(b + c - a)(c + a - b)$.

$F(b, b, c) = (b - c)(c + b - b)(b + b - c) + (c - b)(b + b - c)(b + c - b) + (b - b)(b + c - b)(c + b - b)$

$= (b - c)\, c\, (2b - c) + (c - b)(2b - c)\, c = c(b - c)(2b - c) - c(b - c)(2b - c) = 0$

$\therefore$ $(a - b)$ is a factor of $F(a, b, c)$ by Factor Theorem.

Since $F(a, b, c)$ is cyclic, $(a - b)(b - c)(c - a)$ is a factor of $F(a, b, c)$ .

Deg $[F(a, b, c)] = 3$, $\deg[(a - b)(b - c)(c - a)] = 3$ , the remaining factor must be of degree 0 : $k$.

$F(a, b, c) = k(a - b)(b - c)(c - a)$ …..(1)

Comparing coefficients of $a^2b$ on both sides of (1), we have $k = -4$.

$F(a, b, c) = -4(a - b)(b - c)(c - a) = 4(b - a)(c - b)(a - c)$ .

**4.** Let $P(x) = x^3 + 3x^2 - 2$. Since $x = -1$ is a root of $P(x)$, by division $P(x) = (x + 1)(x^2 + 2x - 2) = 0$

$\therefore$ a, b are roots of $x^2 + 2x - 2 = 0 \Rightarrow a + b = -2$ …. (1), $ab = -2$ …. (2)

Let $f(x) = px^2 + qx + r$ $\therefore f(1) = p + q + r = 1$ ….(3)

$f(a) = pa^2 + qa + r = b$ ….(4)

$f(b) = pb^2 + qb + r = a$ ….(5)

(4) – (5), $p(a + b)(a - b) + q(a - b) = b - a \Rightarrow p(a + b) + q = -1 \Rightarrow 2p - q = 1$ (by (1)) ….(6)

(4) + (5), $p[(a + b)^2 - 2ab] + q(a + b) + 2r = a + b \Rightarrow 8p - 2q + 2r = -2$ (by (1),(2))

$\Rightarrow 4p - q + r = -1$ ….(7)

Solving (3), (6), (7), $p = 4$, $q = 7$, $r = -10$. $\therefore f(x) = 4x^2 + 7x - 10$ is the required polynomial.

**5.** **(** $\Rightarrow$ **)** If $f(x) = ax^3 + bx^2 + cx + d$ is a perfect cube, then

$f(x) = ax^3 + bx^2 + cx + d = a(x + e)^3 = a(x^3 + 3ex^2 + 3e^2x + e^3)$

Compare coefficients, we have $b = 3ae$, $c = 3ae^2$, $d = ae^3$

$\therefore b^3 = (3ae)^3 = 27a^3e^3 = 27a^2(ae^3) = 27a^3d$

and $c^3 = (3ae^2)^3 = 27a^3e^6 = 27a(ae^3)^2 = 27ad^2$.

**(** $\Leftarrow$ **)** If $b^3 = 27a^2d$ and $c^3 = 27ad^2$, then

$$f(x) = ax^3 + bx^2 + cx + d = ax^3 + 3a^{\frac{2}{3}}d^{\frac{1}{3}}x^2 + 3a^{\frac{1}{3}}d^{\frac{2}{3}}x + d = \left(a^{\frac{1}{3}}x + d^{\frac{1}{3}}\right)^3$$

**6.** **(a)** $S(n) = 1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots + n(n+1)$ ….(1)

$S(n) = A + Bn + Cn^2 + Dn^3 + En^4 + \ldots$ ….(2)

From (2), $S(n+1) - S(n) = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 - (A + Bn + Cn^2 + Dn^3 + En^4) + \ldots$

$= (B + C + D + E) + (2C + 3D + 4E)n + (3D + 6E)n^2 + 4En^3 + \ldots$ ….(3)

From (1) $S(n+1) - S(n) = (n + 1)(n + 2) = 2 + 3n + n^2$ ….(4)

Apply the Principle of Undetermined Coefficients, we have, from (3) and (4),

$B + C + D + E = 2$ ….(5)

$2C + 3D + 4E = 3$ ….(6)

$3D + 6E = 1$ ….(7)

$4E = 0$ ….(8)

From (8), $E = 0$ (and the coefficients of higher terms in n are zero)

Solving (5), (6), (7), $D = 1/3$, $C = 1$, $B = 2/3$.

Subst. the values of B, C, D, E in (2), $S(n) = A + \frac{2}{3}n + n^2 + \frac{1}{3}n^3$

$S(1) = A + \frac{2}{3} + 1 + \frac{1}{3} = A + 2$ ….(9) and from (1), $S(1) = 1\times2 = 2$ ….(10)

$\therefore A = 0$ $\therefore S(n) = \frac{2}{3}n + n^2 + \frac{1}{3}n^3 = \frac{1}{3}n(n+1)(n+2)$

**(b)** $S(n) = 1^3 + 3^3 + 5^3 + 7^3 + \dots$ ….(1)

$S(n) = A + Bn + Cn^2 + Dn^3 + En^4 + \dots$ ….(2)

From (2), $S(n+1) - S(n) = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 - (A + Bn + Cn^2 + Dn^3 + En^4\ ) + \dots$

$= (B + C + D + E) + (2C + 3D + 4E)n + (3D + 6E)n^2 + 4En^3 + \dots$ ….(3)

From (1) $S(n+1) - S(n) = (2n + 1)^3 = 8n^3 + 12n^2 + 6n = 1$ ….(4)

Apply the Principle of Undetermined Coefficients, we have, from (3) and (4),

$B + C + D + E = 1$ ….(5)

$2C + 3D + 4E = 6$ ….(6)

$3D + 6E = 12$ ….(7)

$4E = 8$ ….(8)

The coefficients of terms higher $n^4$ are zero and is not calculate here.

Solving (5), (6), (7), $E = 2$, $D = 0$, $C = -1$, $B = 0$.

Subst. the values of B, C, D, E in (2), $S(n) = A - n^2 + 2n^4$ .

$S(1) = A - 1 + 2 = A + 1$ ….(9) and from (1), $S(1) = 1^3 = 1$ ….(10)

$\therefore$ $A = 0$ $\therefore$ $S(n) = - n^2 + 2n^4 = n^2 (2n^2 - 1)$ .

**7.** Let $f(x) = x^3 + ax^2 + 11x + 6$ , $g(x) = x^3 + bx^2 + 14x + 8$ with a common factor of the form $x^2 + px + q$.

$\therefore$ $f(x) = (x^2 + px + q)(x - \alpha) = x^3 + (p - \alpha)x^2 + (q - \alpha p)x - \alpha q$

$g(x) = (x^2 + px + q)(x - \beta)= x^3 + (p - \beta\ )x^2 + (q - \beta p)\ x - \beta q$

Comparing coefficients,

| | | |
|---|---|---|
| $a = p - \alpha$ ….(1) | (5)$\downarrow$(1), $a = p + 6/q$ ….(7) | (11)=(12), $8(11 - q) = 6(14 - q)$ |
| $b = p - \beta$ ….(2) | (6)$\downarrow$(2), $b = p + 8/q$ ….(8) | $\therefore$ $q = 2$ ….(13) |
| $11 = q - \alpha p$ ….(3) | (5) $\downarrow$(3), $11 = q + 6p/q$ ….(9) | (13) $\downarrow$(11), $p = 3$ ….(14) |
| $14 = q - \beta p$ ….(4) | (6) $\downarrow$(4), $14 = q + 8p/q$ ….(10) | (13)(14) $\downarrow$(1), $a = 6$ |
| $-6 = \alpha q$ ….(5) | From (9), $p/q = (11 - q)/6$ ….(11) | (13)(14) $\downarrow$(2), $b = 7$ |
| $-8 = \beta q$ ….(6) | From (10), $p/q = (14 - q)/8$ ….(12) | |

**8.** Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{i=0}^{n} b_i x^i$ , we may assume that the degree of $f(x)$ and $g(x)$ are both equal to n , if not, we can put some $a_i$ or $b_i = 0$ to make the degree the same.

We use induction on n . Let $P(n) : f^2 + x\, g^2 \equiv 0$ , deg f = deg g = n $\Rightarrow f \equiv 0, g \equiv 0$.

For P(0), $f = a_0$ , $g = b_0$, $f^2 + x\, g^2 = a_0^2 + x\, b_0^2 \equiv 0 \Rightarrow$ $a_0^2 = b_0^2 = 0 \Rightarrow a_0 = b_0 = 0$

$\therefore$ $f \equiv 0, g \equiv 0$. $\therefore$ P(0) is true .

Assume P(k) is true for some $k \in \mathbf{N}$ . i.e. $f^2 + x\, g^2 \equiv 0$ , deg f = deg g = k $\Rightarrow f \equiv 0, g \equiv 0$. ….(*)

For P(k + 1), $f^2 + x\, g^2 = \left(\sum_{i=0}^{k+1} a_i x^i\right)^2 + x\left(\sum_{i=0}^{k+1} b_i x^i\right)^2 \equiv 0$

Put x = 0, we have $a_0 = 0$, $f^2 + x\, g^2 = \left(\sum_{i=1}^{k+1} a_i x^i\right)^2 + x\left(\sum_{i=0}^{k+1} b_i x^i\right)^2 \equiv 0$

$\therefore \quad x^2\left(\sum_{i=1}^{k} a_i x^{i-1}\right)^2 + x\left(\sum_{i=0}^{k+1} b_i x^i\right)^2 \equiv 0 \Rightarrow x\left(\sum_{i=1}^{k} a_i x^{i-1}\right)^2 + \left(\sum_{i=0}^{k+1} b_i x^i\right)^2 \equiv 0$

Put $x = 0$, we have $b_0 = 0$, $f^2 + x g^2 = x\left(\sum_{i=1}^{k} a_i x^{i-1}\right)^2 + x^2\left(\sum_{i=1}^{k} b_i x^{i-1}\right)^2 \equiv 0$

$\therefore \quad f^2 + x g^2 = \left(\sum_{i=1}^{k} a_i x^{i-1}\right)^2 + x\left(\sum_{i=1}^{k} b_i x^{i-1}\right)^2 \equiv 0$. where the degrees of f and g are $k$.

By (*), $f \equiv 0, g \equiv 0$. $\therefore$ P(k) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N} \cup \{0\}$.

**9.** $s_1(x_1, x_2, x_3, x_4)$ = coeff. of $y^3 = x_1 + x_2 + x_3 + x_4$.

$s_2(x_1, x_2, x_3, x_4)$ = coeff. of $y^2 = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3$.

$s_3(x_1, x_2, x_3, x_4)$ = coeff. of $y^1 = x_1 x_2 x_3 + x_1 x_2 x_3 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$.

$s_4(x_1, x_2, x_3, x_4)$ = coeff. of $y^0 = x_1 x_2 x_3 x_4$.

**10.** **(a)** $f(x) = \sum_{r=1}^{3n} a_r x^r$, $\omega^3 = 1$, $1 + \omega + \omega^2 = \frac{1-\omega^3}{1-\omega} = \frac{1-1}{1-\omega} = 0$, $1 + \omega^r + \omega^{2r} = \begin{cases} 0 & \text{if } r \neq 3k \\ 3 & \text{if } r = 3k \end{cases}$, $k \in \mathbf{N}$.

$$f(x) + f(\omega x) + f(\omega^2 x) = \sum_{r=1}^{3n} a_r x^r + \sum_{r=1}^{3n} a_r \omega^r x^r + \sum_{r=1}^{3n} a_r \omega^{2r} x^r = \sum_{r=1}^{3n} \left(1 + \omega^r + \omega^{2r}\right) a_r x^r = 3\sum_{r=1}^{n} a_{3r} x^{3r}$$

**(b)** $f(x) + \omega f(\omega x) + \omega^2 f(\omega^2 x)$

$$= \sum_{r=1}^{3n} a_r x^r + \omega\sum_{r=1}^{3n} a_r \omega^r x^r + \omega^2\sum_{r=1}^{3n} a_r \omega^{2r} x^r = \sum_{r=1}^{3n}\left(1 + \omega^{r+1} + \omega^{2(r+1)}\right) a_r x^r = 3\sum_{r=1}^{n} a_{3r-1} x^{3r-1}$$

$f(x) + \omega^2 f(\omega x) + \omega(\omega^2 x)=$

$$\sum_{r=1}^{3n} a_r x^r + \omega^2\sum_{r=1}^{3n} a_r \omega^r x^r + \omega\sum_{r=1}^{3n} a_r \omega^{2r} x^r = \sum_{r=1}^{3n}\left(1 + \omega^{r+2} + \omega^{2r+1}\right) a_r x^r = 3\sum_{r=1}^{n} a_{3r-2} x^{3r-2}$$

11. (Guass Lemma) Let $b(x) = b_0 + b_1 x + \ldots + b_{q-1}x^{q-1} + x^q$, $c(x) = c_0 + c_1 x + \ldots + c_{r-1}x^{r-1} + x^r$.

The fractions $b_i$ have a least common denominator $\beta > 0$.

Put $b_i = \frac{\beta_i}{\beta}$, then $\beta_0, \beta_1, \ldots, \beta_{q-1}, \beta_q \in Z$ and have non common integral factors except $\pm 1$.

Let $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}, \gamma_r \in Z$ be similarly defined.

**(i)** If $\beta = \gamma = 1$, there is nothing to prove.

**(ii)** Accordingly, suppose that $\beta\gamma > 1$, then $\beta\gamma\, a(x) = B(x)\, C(x)$ where

$B(x) = \beta_0 + \beta_1 x + \ldots + \beta_{q-1}x^{q-1} + \beta x^q$, $C(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_{r-1}x^{r-1} + \gamma x^r$.

Let p be a prime factor of $\beta\gamma$; then p divides each coeff. in $B(x)\, C(x)$.

At least one coeff. in $B(x)$ is not divisible by p ; among such coeffs. let $\beta_i$ be the one with highest suffix. Similarly define $\gamma_j$.

The coeff. of $x^{i+j}$ in $B(x)C(x)$ is $\beta_{i+j}\gamma_0 + \ldots + \beta_{i+1}\gamma_{j-1} + \beta_i\gamma_j + \beta_{i-1}\gamma_{j+1} + \ldots + \beta_0\gamma_{i+j}$

Since $\beta$ ; $\beta_0, \ldots, \beta_{i+1}$ and $\gamma$; $\gamma_{j+1}, \ldots, \gamma_{i+j}$ are all divisible by p, but $\beta_i\gamma_j$ is not divisible by p.

The supposition $\beta\gamma > 1$ thus leads to a contadiction.

**12. (a)** $a(x)f(x) + g(x) = 0$ ; $g(x) = 0$ or $Deg[g(x)] < Deg\,[a(x)]$

$\Rightarrow g(x) = - a(x)f(x) \Rightarrow g(x)$ has a factor $a(x) \Rightarrow Deg[g(x)] \geq Deg\,[a(x)]$

So, if $Deg[g(x)] < Deg\,[a(x)]$ , there is a contradiction.

$\therefore$ $g(x) = 0$ and $- a(x)f(x) = 0 \Rightarrow f(x) = 0$

**(b)** $b(x) = a(x)q(x) + r(x)$ ….(1) $Deg[r(x)] < Deg[a(x)]$

$b(x) = a(x)Q(x) + R(x)$ ….(2) $Deg[R(x)] < Deg[a(x)]$

(1) – (2), $a(x)\,[q(x) - Q(x)] + [r(x) - R(x)] = 0$

By (a), $q(x) - Q(x) = 0$ and $r(x) - R(x) = 0$. $\therefore q(x) = Q(x)$, $r(x) = R(x)$ .

**13. (a)** By Division Algorithm,

$f(x) = g(x)\, q_0(x) + r_0(x)$, $Deg[g(x)] > Deg[r_0(x)]$

$g(x) = r_0(x)\, q_1(x) + r_1(x)$, $Deg[r_0(x)] > Deg[r_1(x)]$

$r_0(x) = r_1(x)\, q_2(x) + r_2(x)$, $Deg[r_1(x)] > Deg[r_2(x)]$

:: :: :: :: :: ::

$r_{n-2}(x) = r_{n-1}(x)\, q_n(x) + r_n(x)$ $Deg[r_{n-1}(x)] > Deg[r_n(x)]$

$r_{n-1}(x) = r_n(x)\, q_{n+1}(x)$

Let $P(n)$ : $\exists$non-zero polynomials $M_i(x), N_i(x)$ such that $r_i(x) = M_i(x)f(x) + N_i(x)g(x)$

For $P(0)$, $r_0(x) = f(x) - q_0(x)g(x)$, $\therefore$ P(1) is true.

For $P(1)$, $r_1(x) = g(x) - q_1(x)r_0(x) = g(x) - q_1(x)\,[f(x) - q_0(x)g(x)] = - q_1(x)\, f(x) + [1 + q_0(x)\, q_1(x)]g(x)$

$\therefore$ P(1) is true.

Assume $r_i(x) = M_i(x)f(x) + N_i(x)g(x)$ is true $\forall$ $i < k$.

For $P(k)$, $r_k(x) = r_{k-2}(x) - r_{k-1}(x)\, q_k(x)$

$= M_{k-2}(x)f(x) + N_{k-2}(x)g(x) - [M_{k-1}(x)f(x) + N_{k-1}(x)g(x)]\, q_k(x)$

$= [M_{k-2}(x) - M_{k-1}(x)\, q_k(x)]\, f(x) + [N_{k-2}(x) - N_{k-1}(x)\, q_k(x)]\, g(x)$

$= M_k(x)f(x) + N_k(x)g(x)$ $\therefore$ P(k) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N} \cup \{0\}$.

In particular, $r_n\,(x) = M\,(x)f(x) + N\,(x)g(x)$,

since by Euclidean Algorithm, $r_n(x)$ is the H.C.F. of $f(x), g(x)$.

**(b)** Since $f(x), g(x)$ are relatively prime, H.C.F. of $f(x), g(x) = 1$, by part (a), we have

$1 = M(x)f(x) + N\,(x)g(x)$

$\therefore$ $M(x)\, f(x) = 1 - N(x)\, g(x)$ ….(1)

$N(x)\, g(x) = 1 - M(x)\, f(x)$ ….(2)

If $R(x)f(x) = S(x)g(x)$, then $R(x)N(x)f(x) = S(x)\, N(x)g(x) = S(x)\,[1 - M(x)\, f(x)]$ , by (2)

$\therefore$ $S(x) = f(x)\,[R(x)N(x) + S(x)\, M(x)]$ and $f(x)$ divides $S(x)$.

If $S(x)g(x) = R(x)f(x)$, then $S(x)M(x)g(x) = R(x)M(x)f(x) = R(x)\,[1 - N(x)\, g(x)]$, by (1)

$\therefore$ $R(x) = g(x)\,[S(x)M(x) + R(x)\, N(x)]$ and $g(x)$ divides $R(x)$.

**14.** **(a)** $a^2(b-c)+b^2(c-a)+c^2(a-b)=0 \Rightarrow -(a-b)(b-c)(c-a)=0 \Rightarrow a=b \vee b=c \vee c=a$

W.l.o.g. let $a=b$, $(a^n-b^n)(b^n-c^n)(c^n-a^n)=(b^n-b^n)(b^n-c^n)(c^n-b^n)=0$.

**(b)** $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \Rightarrow (a+b+c)(bc+ca+ab)=abc \Rightarrow (a+b)(b+c)(c+a)=0$

$\Rightarrow a=-b \vee b=-c \vee c=-a$, W.l.o.g. let $a=-b$,

$$\therefore \frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{(-b)^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{c^{2n+1}}=\frac{1}{(-b+b+c)^{2n+1}}=\frac{1}{(a+b+c)^{2n+1}}$$

**(c)** $\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=1 \Rightarrow (a+b-c)(b+c-a)(c+b-a)=0$

W.l.o.g. let $a+b-c=0$,

$$\frac{b^2+c^2-a^2}{2bc}=\frac{(b-c)^2-a^2}{2bc}+1=\frac{(b-c+a)(b-c-a)}{2bc}+1=0+1=1$$

Similarly, $\frac{c^2+a^2-b^2}{2ca}=1$, $\frac{a^2+b^2-c^2}{2ab}=-1$

$$\left(\frac{b^2+c^2-a^2}{2bc}\right)^{2n+1}+\left(\frac{c^2+a^2-b^2}{2ca}\right)^{2n+1}+\left(\frac{a^2+b^2-c^2}{2ab}\right)^{2n+1}=1^{2n+1}+1^{2n+1}+(-1)^{2n+1}=1$$

**15.** **(a)** Let $f(a,b,c)=ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)$

$f(b,b,c)=b^2(b^2-b^2)+bc(b^2-c^2)+cb(c^2-b^2)=bc(b^2-c^2)-bc(b^2-c^2)=0$

By Factor theorem, $a-b$ is a factor of $f$.

Since $f$ is cyclic, $(a-b)(b-c)(c-a)$ is a factor of $f(a,b,c)$.

Deg $[f(a,b,c)]=4$, $\deg[(a-b)(b-c)(c-a)]=3$, the remaining factor must be of deg 1 : $k(a+b+c)$.

$f(a,b,c)=k(a+b+c)(a-b)(b-c)(c-a)$ …..(1)

Comparing coefficients of $ab^3$ on both sides of (1), we have $k=-1$.

$F(a,b,c)=-(a+b+c)(a-b)(b-c)(c-a)=(a+b+c)(b-a)(c-b)(a-c)$.

**(b)** Let $f(x,y,z)=(x-y)^5+(y-z)^5+(z-x)^5$

$f(y,y,z)=(y-y)^5+(y-z)^5+(z-y)^5=(y-z)^5-(y-z)^5=0$

By Factor theorem, $x-y$ is a factor of $f$. Since $f$ is cyclic, $(x-y)(y-z)(z-x)$ is a factor of $f(x,y,z)$.

Deg $[f(x,y,z)]=5$, $\deg[(x-y)(y-z)(z-x)]=3$, the remaining factor must be of deg 2

$f(x,y,z)=[k_1(x^2+y^2+z^2)+k_2(xy+yz+zx)](x-y)(y-z)(z-x)$ …..(1)

Comparing coeff. of $x^4y$, $k=5$ ; Comparing coeff. of $x^3y^2$, $k=-5$

$f(x,y,z)=5(x^2+y^2+z^2-xy-yz-zx)(x-y)(y-z)(z-x)$

**(c)** Let $f(a,b,c)=(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$

$f(0,b,c)=(b+c)^4-(b+c)^4-c^4-b^4+b^4+c^4=0$

By Factor theorem, $a$ is a factor of $f$. Since $f$ is cyclic, $abc$ is a factor of $f$.

Deg $[f(a,b,c)]=4$, $\deg[abc]=3$, the remaining factor must be of deg 1 : $k(a+b+c)$.

$f(a, b, c) = kabc\,(a+b+c)$ …..(1)

$f(1, 1, 1) = 3k = (1 + 1 +1)^4 - 2^4 - 2^4 - 2^4 + 1 + 1 + 1 = 36 \quad \Rightarrow \quad k = 12$

$\therefore \quad f(a, b, c) = 12abc\,(a+b+c)$

**16.** Let $f(x) = (x - a)(x - b)\,q(x) + (Ax + B)$, by Division Algorithm

$\therefore \quad f(a) = Aa + B, \quad f(b) = Ab + B$

Solving , $A = \dfrac{f(a) - f(b)}{a - b}$, $B = \dfrac{af(b) - bf(a)}{a - b}$

$\therefore$ Remainder $= Ax + B = \dfrac{f(a) - f(b)}{a - b}x + \dfrac{af(b) - bf(a)}{a - b}$.

**17.** Let $f(x) = (x^2 + x + 1)^n - (x^2 - x - 1)^n$.

$f(-1) = [(-1)^2 + (-1) + 1]^n + [(-1)^2 - (-1) + 1]^n = 1 - 1 = 0$

$\therefore$ By Factor theorem, $1 + x$ is a factor of $f(x)$ or $(x^2 + x + 1)^n - (x^2 - x - 1)^n$ is divisible by $1 + x$.

Put $x = 10$, $f(10) = (10^2 + 10 + 1)^n - (10^2 - 10 - 1)^n = 111^n - 89^n$.

Also, $1 + x = 1 + 10 = 11$, $\therefore$ $111^n - 89^n$ is divisible by 11 .

**18.** **(a)** Let $f(x) = x^6 + 3x^4 - 7 = a(x^2 + 1)^3 + b(x^2 + 1)^2 + c\,(x^2 + 1) + d$

Compare coeff. of $x^6$-term, $a = 1$,

Compare coeff. of $x^4$-term, $3 = 3a + b = 0 \quad \Rightarrow \quad b = 0$,

Compare coeff. of $x^2$-term, $0 = 3a + 2b + c \quad \Rightarrow \quad c = -3$,

Compare coeff. of constant-term, $-7 = a + b + c + d \quad \Rightarrow \quad d = -5$

$f(x) = x^6 + 3x^4 - 7 = (x^2 + 1)^3 - 3(x^2 + 1) - 5$

**(b)** $P(x) = x^8 + x^7 + 6x^6 + 3x^5 + 12x^4 + 4x^2 - 7x - 13$

$= (x^2 + 1)^4 + (x^6 + 3x^4 - 7)(x + 2)$

$Q(x) = (x + 2)\,(x^2 + 1)^4$

$$\frac{P(x)}{Q(x)} = \frac{(x^2+1)^4 + (x^6+3x^4-7)(x+2)}{(x^2+1)^4(x+2)} = \frac{1}{x+2} + \frac{x^6+3x^4-7}{(x^2+1)^4} = \frac{1}{x+2} + \frac{(x^2+1)^3 - 3(x^2+1) - 5}{(x^2+1)^4} \text{, by (a)}$$

$$= \frac{1}{x+2} + \frac{1}{x^2+1} - \frac{3}{(x^2+1)^3} - \frac{5}{(x^2+1)^4}$$